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WEAK CONVERGENCE OF A SEQUENCE OF QUEUEING AND STORAGE
PROCESSES TO A SIN. (U) MASSACHUSETTS UNIV AMHERST DEPT
OF MATHEMATICS AND STATISTICS. W A ROSENKRANTZ NOV 84

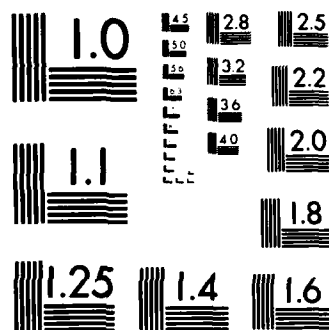
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WEAK CONVERGENCE OF A SEQUENCE OF QUEUEING AND
STORAGE PROCESSES TO A SINGULAR DIFFUSION.

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1. INTRODUCTION

It has been known for a long time that heavy traffic limit theorems in queueing theory are but a special case of the so-called diffusion approximation in Physics and Genetics. Take for example Kingman's (1962) heavy traffic approximation for the stationary waiting time distribution for a sequence of GI/GI/1 queues $Q(\alpha)$ depending on a parameter α . Denote the waiting time, excluding service, of the n th customer by $W(n, \alpha)$ and let $U(n, \alpha) = S(n, \alpha) - T(n, \alpha)$ where $S(n, \alpha)$ = service time of the n th customer and $T(n, \alpha)$ = inter arrival time between the n th and $(n+1)$ st customer and assume $E(U(n, \alpha)) = -\alpha$, variance of $U(n, \alpha) = \sigma^2$, $\alpha > 0$. Then we have the following Theorem 1 (Kingman (1962)):

$$\lim_{n \rightarrow \infty} P((\alpha/\sigma)W(n, \alpha) \leq x) = 1 - \exp(-2x), \quad 0 \leq x < \infty, \quad \text{provided} \quad \lim_{n \rightarrow \infty} \alpha^2 n = \infty.$$

Somewhat later Kingman (1965) presented a more elegant but heuristic proof of this result which justifies referring to such a theorem as a diffusion approximation. It is worthwhile sketching the heuristic proof of Theorem 1 here, referring the reader to Rosenkrantz (1980) for a rigorous proof as well as an estimate of the rate of convergence. To begin with, one notes that

$$(1.1) \quad F_{n, \alpha}(x) = P((\alpha/\sigma)W(n, \alpha) \leq x) = P(\sup_{0 \leq t \leq \alpha^2 n} y_{n, \alpha}(t) \leq x)$$

where $y_{n, \alpha}(t)$ is a certain stochastic process with continuous paths. One can then show, formally at least, that

$$(1.2) \quad \lim_{n \rightarrow \infty, \alpha \rightarrow 0} y_{n, \alpha}(t) = y(t)$$

where $y(t) = w(t) - t$. Here $w(t)$ is the standard 1-dimensional Wiener process and so $y(t)$ is the Wiener process with negative drift. It follows at once from (1.2) that

$$(1.3) \quad \lim_{n \rightarrow \infty, \alpha \rightarrow 0} P(\sup_{0 \leq t \leq \alpha^2 n} y_{n, \alpha}(t) \leq x) = P(\sup_{0 \leq t < \infty} y(t) \leq x)$$

and an easy calculation, see e.g. Karlin-Taylor (1975), p.361, yields the result that $P(\sup_{0 \leq t < \infty} y(t) \leq x) = 1 - \exp(-2x)$, $0 \leq x < \infty$.

Another and simpler example of a heavy traffic limit theorem is the following: let $N_n(t)$ denote the queue size of an M/M/1 queue with arrival rate λ_n , mean

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service time distribution μ_n^{-1} and traffic intensity $\rho_n = \lambda_n / \mu_n$. Assume $\lambda_n = \mu_n - \delta n^{-1/2}$ for some $\delta > 0$, so $0 < \rho_n < 1$ and denote by σ_n^2 the variance of the service time distribution which in this case equals μ_n^{-2} .

THEOREM 2: Assume $\lambda = \lim_{n \rightarrow \infty} \lambda_n = \lim_{n \rightarrow \infty} \mu_n = \mu$ so $\lim_{n \rightarrow \infty} \rho_n \uparrow 1$, and $\lim_{n \rightarrow \infty} \sigma_n^2 = \sigma^2$; then $\lim_{n \rightarrow \infty} X_n(nt)/\sqrt{n} = y(t)$ where $y(t)$ is the Wiener process on $R^+ = [0, \infty)$ with variance $\lambda + \sigma^2 \mu^3$, negative drift δ and reflected at the origin. Theorem 2 has been extended in many ways and by many authors including Iglehart and Whitt. The survey article by Whitt (1974) is a useful reference for the reader interested in these developments.

In each of the heavy traffic limit theorems cited above the limit process has turned out to be the Wiener process with a negative drift satisfying, where appropriate, a reflecting boundary condition. Recently Yamada (1982) has given a diffusion approximation for a sequence of storage processes $X_n(t)$ where the limit process $Y(t)$ is no longer a Wiener process with a negative drift but is instead a Bessel process with negative drift. This result is of more than routine interest. It shows for example that the set of possible limit processes that can occur in queueing and storage theory is a much larger class than Theorems 1, 2 and the survey article by Whitt (1974) would lead us to believe existed. In addition Yamada's theorem (a precise version of which will be stated below as Theorem 3) offers a challenge to the traditional methods by which such limit theorems are usually proved. In particular, neither the Trotter-Kato-Kurtz method of Kurtz (1969) nor the martingale method of Papadimitriou, Stroock and Varadhan (1977) are directly applicable to this limit theorem because of some nontrivial technical problems of independent interest and the solutions of which are also of independent interest. It is the purpose of this paper to give a new and simpler proof of Yamada's theorem using some results due to Brezis, Rosenkrantz and Singer, with an appendix by P. D. Lax, (1971) which, restated in the more modern terminology of today, implies that the martingale problem for the operator corresponding to the Bessel process with drift has a unique solution - see Stroock-Varadhan (1979) and Ikeda-Watanabe (1981) for a general discussion of these ideas. It turns out however that the estimates we needed to make the martingale methods work already imply the strong convergence of the semigroups in the sense of Trotter-Kato - see Theorem 4 below. These as well as other results from Functional Analysis are collected in an appendix. We shall also use the standard notations: $C_0(R^+) = \{f: f \text{ bounded and continuous on } R^+ = [0, \infty) \text{ and } \lim_{x \rightarrow \infty} f(x) = 0\}$, $f^{(k)}(x) = k^{\text{th}} \text{ derivative of } f$, $C_0^k(R^+) = \{f \in C_0(R^+): f^{(l)} \in C_0(R^+), 1 \leq l \leq k\}$. We make $C_0(R^+)$ into a Banach space in the usual way by giving it the norm $\|f\| = \sup_{0 \leq x < \infty} |f(x)|$. The symbol \blacksquare denotes the end of a proof.

2. STATEMENT AND PROOF

Let $X(t)$ denote a process with release rate λ to be a compound Poisson process. The cumulative distribution function $F(t)$ has a stochastic integral representation (Pinsky (1972) have a stochastic integral representation for a compound Poisson process with release rate λ and service time distribution μ).

$$(2.1) \quad X(t) = X(0) + \sum_{i=1}^{N_\lambda(t)} S_i \quad \text{where } N_\lambda(t) \text{ is a Poisson process with rate } \lambda \text{ and } S_i \text{ is a non-negative, non-decreasing process.}$$

now on we also assume $\rho = \lambda \mu_1$ and $k = \lambda \mu_2$.

Following Yamada

$$(2.2) \quad X_n(t) = X(t) + \sum_{i=1}^{N_{\lambda_n}(t)} S_i^n$$

$$A_n(t) = \sum_{i=1}^{N_{\lambda_n}(t)} S_i^n$$

$$(2.3) \quad \bar{r}_n \geq \rho_n$$

$$(2.4) \quad x(\bar{r}_n - r_n)$$

$$(2.5) \quad \lim_{n \rightarrow \infty} n^{1/2} (r_n - \rho_n)$$

$$(2.6) \quad \lim_{n \rightarrow \infty} k_n = k$$

$$(2.7) \quad \sup_{n, x \geq 0} x(\bar{r}_n - r_n)$$

$$(2.8) \quad X_n(0) = x$$

$$(2.9) \quad \lim_{c \rightarrow \infty} \int_{\{y > c\}} f(y) dy = 0$$

From these conditions we can conclude that $\{X_n(t)\}$ is a bounded sequence of processes. This implies $\{X_n(t)\}$ is a bounded sequence of processes.

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Chief, Technical Information Division

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2. STATEMENT AND PROOF OF YAMADA'S DIFFUSION APPROXIMATION.

Let $X(t)$ denote the content of a dam at time t (also called a storage process) with release rate $r(x)$ and random cumulative input $A(t)$ which is assumed to be a compound Poisson process. The jump rate λ is assumed to be finite and the cumulative distribution of the size of the jump is denoted by $F(y)$. Cinlar-Pinsky (1972) have shown that $X(t)$ may be realized as the unique solution of the stochastic integral equation

$$(2.1) \quad X(t) = X(0) - \int_0^t r(X(s)) ds + A(t), \text{ where}$$

$$A(t) = \sum_{i=1}^{N_\lambda(t)} S_i \text{ where the } S_i \text{ are i.i.d. with common distribution } F \text{ and } N_\lambda(t)$$

is a Poisson process with intensity λ . The release rate $r(x)$ is assumed to be a non-negative, non-decreasing function with domain $R^+ = [0, \infty)$, $r(0) = 0$. From

now on we also assume that $\bar{r} = \lim_{x \rightarrow \infty} r(x)$ is finite. We set $\mu_1 = \int_0^\infty y dF(y)$,

$$\rho = \lambda \mu_1 \text{ and } k = \sqrt{\lambda \mu_2}.$$

Following Yamada (1982) we make the following hypotheses:

$$(2.2) \quad X_n(t) = X_n(0) - \int_0^t r_n(X_n(s)) ds + A_n(t), \quad n = 1, 2, \dots$$

is a sequence of storage processes with release rates $r_n(x)$,

$$A_n(t) = \sum_{i=1}^{N_{\lambda_n}(t)} S_i^n, \quad P(S_i^n \leq y) = F_n(y) \text{ satisfying the normalization conditions:}$$

$$(2.3) \quad \bar{r}_n \geq \rho_n, \quad \rho_n = \lambda_n \mu_1^n, \quad \mu_1^n = \int_0^\infty y dF_n(y)$$

$$(2.4) \quad x(\bar{r}_n - r_n(x)) \rightarrow c < \infty, \text{ as } x \rightarrow \infty, n \rightarrow \infty$$

$$(2.5) \quad \lim_{n \rightarrow \infty} n^{1/2}(\bar{r}_n - \rho_n)/k_n = d,$$

$$(2.6) \quad \lim_{n \rightarrow \infty} k_n = k > 0, \quad k_n^2 = \lambda_n \mu_2^n$$

$$(2.7) \quad \sup_{n, x \geq 0} x(\bar{r}_n - r_n(x)) = M < \infty$$

$$(2.8) \quad X_n(0) = x_n, \quad \lim_{n \rightarrow \infty} x_n/k_n \sqrt{n} = x.$$

$$(2.9) \quad \lim_{c \rightarrow \infty} \int_{\{y > c\}} y^2 dF_n(y) = 0 \text{ uniformly in } n.$$

From these conditions it is easy to see that each of the following sequences is bounded: $\{\mu_2^n\}$, $\{\mu_1^n\}$, $\{\lambda_n\}$, $\{\rho_n\}$ and $\{\bar{r}_n\}$. For example (2.9) implies that

$\{\mu_2^n\}$ is a bounded sequence and *a fortiori* so is $\{\mu_1^n\}$. This together with (2.6) implies $\{\lambda_n\}$ is bounded and the other statements are proved in a similar fashion.

THEOREM 3 (Yamada): Set $Y_n(t) = X_n(nt)/k_n\sqrt{n}$ and assume conditions (2.3) through (2.9) hold and that $\lim_{n \rightarrow \infty} Y_n(0) = x$. Then $Y_n(t)$ converges weakly to a Bessel process with negative drift $Y(t)$, starting at x . $Y(t)$ is a (Markov) diffusion process on $R^+ = [0, \infty)$ whose infinitesimal generator is given by

$$(2.10) \quad Gf(x) = (1/2)f''(x) + (c/k^2)(f'(x)/x) - df'(x).$$

Remarks: This is not the form in which Yamada states his theorem. Specifically, he shows that $Y(t) = \sqrt{Z(t)}$ where $Z(t)$ is the unique solution to the stochastic integral equation:

$$(2.11) \quad Z(t) = Z(0) + \int_0^t (K - 2d\sqrt{Z(s)})ds + 2 \int_0^t \sqrt{Z(s)}dw(s)$$

where $K = 1 + 2c/k^2$ and w is the standard Wiener process. Thus $Z(t)$ satisfies the stochastic differential equation

$$(2.12) \quad \begin{cases} dZ(t) = (K - 2d\sqrt{Z(t)})dt + 2\sqrt{Z(t)}dw(t) \\ = b(Z(t))dt + a(Z(t))dw(t) \text{ with} \\ b(x) = (K - 2d\sqrt{x}), \quad x \geq 0 \text{ and } a(x) = 2\sqrt{x}. \end{cases}$$

Notice that neither $a(x)$ nor $b(x)$ (when $d \neq 0$) are Lipschitz continuous and so the existence of a unique solution to the stochastic differential equation (2.12) is not a trivial matter. The existence of a unique solution is however a consequence of a more general result due to Okabe and Shimizu (1975). Before proceeding to our own proof let us sketch the idea behind Yamada's proof. He first shows that the processes $Y_n(t)$ are tight in $D[0, T]$ and that if $Y(t)$ is any limit then $Z(t) = Y(t)^2$ solves the martingale problem:

$$(2.13) \quad f(Z(t)) - f(Z(0)) - \int_0^t (K - 2d\sqrt{Z(s)})f'(Z(s))ds - 2 \int_0^t \sqrt{Z(s)}f''(Z(s))ds \text{ is a zero mean martingale for every } f \in C_K^2(R).$$

$C_K^2(R)$ is the set of twice continuously differentiable functions, with compact support. This shows that every weak limit solves the martingale problem (2.13) which, thanks to the results of Okabe-Shimizu, op. cit, is known to have a unique solution. The proof that $Z(t)$ is a solution to the martingale problem (2.13) is almost 5 pages long and the proof that the processes $\{Y_n(t)\}$ form a tight sequence is nearly 6 pages long. It is the purpose of this paper to give an alternative proof of this result which we believe to be easier to follow and is also somewhat shorter. First we shall give a heuristic proof and put in the (tedious) details elsewhere.

We begin by observing that $Y_n(t)$ is for each n a Markov process on the half line $R^+ = [0, \infty)$ with infinitesimal generator G_n given by

$$(2.14) \quad \begin{cases} G_n f(x) = - \\ G_n f(0) = - \\ \text{Here } H_n(v) \end{cases}$$

See for example Cinlar where the operators are given in some detail.

DEFINITION: $D(G) =$ at (2.12)

Later on, in Appendix A, we characterize $D(G)$ is as follows: this was already done in the case $d \neq 0$ is $-df'(x)$ is relative-

$$(2.15) \quad Bf(x) = (1/2) \frac{d}{dx} (f(x) \sqrt{x})$$

in the sense of Kato. In this way we can give a quick proof of

LEMMA 1: For every $f \in D(G)$, convergence is uniform $\sup_n \|G_n f(x)\| < \infty$.

PROOF: Using the Tanaka formula where $R(x, y) = (1/2) |x - y|$

$$n\lambda_n \int_0^\infty [f(x) - f(y)] dy$$

where $|2R(n)| \leq n\lambda_n$ and $\lim_{n \rightarrow \infty} R(n) = 0$. On the other hand

$$(2.16) \quad G_n(f(x)) = - \frac{1}{2} \frac{d}{dx} (f(x) \sqrt{x})$$

since $n\lambda_n \mu_1^n / k_n \sqrt{n} = 1$

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cess) diffusion

$$(2.14) \quad \begin{cases} G_n f(x) = -(\sqrt{n}/k_n) r_n(k_n \sqrt{n} \cdot x) f'(x) + n \lambda \int_0^\infty [f(x+y) - f(x)] dH_n(y) \\ \text{for } x > 0 \quad \text{and} \\ G_n f(0) = n \lambda \int_0^\infty [f(y) - f(0)] dH_n(y). \end{cases}$$

Here $H_n(y) = F_n(k_n \sqrt{n} \cdot y)$.

Specifically, he
the stochastic in-

See for example Cinlar-Pinsky (1972), Harrison-Kesnick (1976) or Rosenkrantz (1981) where the operators G_n and their domains (both strong and weak) are discussed in some detail.

us $Z(t)$ satisfies

DEFINITION: $D(G) = \{f \in C_0^2(\mathbb{R}^+): f'(0) = 0\}$, where the operator G is defined at (2.10).

Later on, in Appendix A, we will show that $D(G)$ is the domain of the strong infinitesimal generator of the semi group $T(t)f(x) = E_x(f(Y(t)))$. Of course, characterizing $D(G)$ is not, in general, an easy matter but in the special case $d = 0$ this was already done by Brezis et al. (1971). The extension of their results to the case $d \neq 0$ is carried out in this paper by showing that the operator $Cf(x) = -df'(x)$ is relatively bounded with respect to the Bessel operator

$$(2.15) \quad Bf(x) = (1/2)f''(x) + (\gamma/x)f'(x), \quad \gamma > -1/2,$$

in the sense of Kato (1976) cf. Appendix A. With these preliminaries out of the way we can give a quick heuristic proof of Yamada's theorem by deriving the

LEMMA 1: For every $f \in D(G)$ and $x > 0$ we have $\lim_{n \rightarrow \infty} G_n f(x) = Gf(x)$; the convergence is uniform on every interval of the form $[\delta, \infty)$, $\delta > 0$ and $\sup_n \|G_n f(x)\| < \infty$.

PROOF: Using the Taylor expansion $f(x+y) - f(x) = f'(x)y + (1/2)f''(x)y^2 + R(x,y)$ where $R(x,y) = (1/2)(f''(\xi(y)) - f''(x))y^2$ and $x \leq \xi(y) \leq x+y$, we see that

$$n \lambda \int_0^\infty [f(x+y) - f(x)] dH_n(y) = n \lambda f'(x) \int_0^\infty y dH_n(y) + (1/2) n \lambda f''(x) \int_0^\infty y^2 dH_n(y) + R(n)$$

where $|2R(n)| \leq n \lambda \int_0^\infty |f''(\xi(y)) - f''(x)| y^2 dH_n(y)$. In a moment we will show that $\lim_{n \rightarrow \infty} R(n) = 0$. On the other hand $\int_0^\infty y dH_n(y) = \mu_1^n/k_n \sqrt{n}$ and $\int_0^\infty y^2 dH_n(y) = \mu_2^n/k_n^2$ so

$$(2.16) \quad G_n f(x) = [-(\sqrt{n}/k_n) r_n(k_n \sqrt{n} \cdot x) + (\sqrt{n}/k_n)] f'(x) + (1/2) f''(x) + R(n),$$

since $n \lambda \mu_1^n/k_n \sqrt{n} = \sqrt{n}/k_n$ and $n \lambda \mu_2^n/k_n^2 = 1$ - see (2.3) and (2.6). Adding and

subtracting the term $(\sqrt{n}/k_n)\bar{r}_n f'(x)$ to the right hand side of (2.16) we obtain

$$G_n f(x) = (\sqrt{n}/k_n)(\bar{r}_n - r_n(k_n \sqrt{n} \cdot x))f'(x) + (\sqrt{n}/k_n)(\rho_n - \bar{r}_n)f'(x) \\ + (1/2)f''(x) + R(n).$$

For $x > 0$ we have $(\sqrt{n}/k_n)(\bar{r}_n - r_n(k_n \sqrt{n} \cdot x))f'(x) = (k_n \sqrt{n} \cdot x/k_n^2)(\bar{r}_n - r_n(k_n \sqrt{n} \cdot x))f'(x)/x$ consequently (2.4)(2.6),(2.7) imply that for $x > 0$ $\lim_{n \rightarrow \infty} (\sqrt{n}/k_n)(\bar{r}_n - r_n(k_n \sqrt{n} \cdot x))f'(x) = (c/k^2)f'(x)/x$ and the convergence is uniform on the interval $[\delta, \infty)$. Hypothesis (2.7) implies that the term is uniformly bounded in n and x . Similarly condition (2.5) implies $\lim_{n \rightarrow \infty} (\sqrt{n}/k_n)(\bar{r}_n - \rho_n)f'(x) = -df'(x)$. Thus the lemma will be proved if we can show that $\lim_{n \rightarrow \infty} R(n) = 0$, where

$$|2R(n)| \leq n\lambda_n \int_0^\epsilon |f''(\xi(y)) - f''(x)| y^2 dH_n(y) + n\lambda_n \int_\epsilon^\infty |f''(\xi(y)) - f''(x)| y^2 dH_n(y).$$

Now for ϵ small enough $|f''(\xi(y)) - f''(x)| < \delta$ and this together with the fact that $n\lambda_n \int_0^\epsilon y^2 dH_n(y) \leq n\lambda_n \int_0^\infty y^2 dH_n(y) = 1$ implies that the first summand in the expression above can be made arbitrarily small. As for the second summand a change of variable yields the formula $n\lambda_n \int_\epsilon^\infty y^2 dH_n(y) = (\lambda_n/k_n^2) \int_{k_n \sqrt{n} \cdot \epsilon}^\infty z^2 dF_n(z)$ which goes to zero by hypothesis (2.9) and the fact that both λ_n and k_n^2 are bounded. \square

It is easy to see that $\lim_{n \rightarrow \infty} G_n f(0) \neq Gf(0)$. Because $Gf(0) = (1/2)f''(0) + (c/k^2)f''(0) - df'(0) = (1/2 + c/k^2)f''(0)$ since $f'(0) = 0$ and $f \in C_0^2(\mathbb{R}^+)$ implies $f''(0) = \lim_{x \rightarrow 0} \frac{f'(x)}{x}$. On the other hand (by (2.14)) $G_n f(0) = n\lambda_n \int_0^\infty (f(y) - f(0))dH_n(y)$ and using a two term Taylor expansion as before we get that

$\lim_{n \rightarrow \infty} G_n f(0) = (1/2)f''(0)$. Thus the only time $G_n f(x)$ converges $Gf(x)$ for all $x \in \mathbb{R}^+$ is in the special case $c = 0$. i.e. when the limiting process $Y(t)$ is the Wiener process with a negative drift reflected at the origin. This phenomenon of convergence of the generators except at certain exceptional points is quite common and occurs even in the example of Theorem 2 - cf. Burman (1979) p.17. Nevertheless, it has been observed by several authors including Papanicolaou, Stroock, Varadhan (1975), Burman (1979) that weak convergence of $Y_n(t)$ to $Y(t)$ can be proved, provided one can show that the occupation time of the exceptional set by the process $Y_n(t)$ can be made arbitrarily small as $n \rightarrow \infty$. In the present context we must estimate $\int_0^t I_{[0, \delta]}(Y_n(s))ds$ which is the occupation time of the set $[0, \delta]$ by the process $Y_n(t)$.

LEMMA 2: Under the such that

$$(2.17) \quad \lim_{n \rightarrow \infty} \sup$$

Setting aside strong convergence

THEOREM 4: Under

$$(2.18) \quad \lim_{n \rightarrow \infty} \|E$$

where the convergence to the proof of characterizing the fined at (2.14).

LEMMA 3: Let G_n storage processes.

Case 1:

$$(2.19)$$

$$(2.20) \quad \text{Case 2:}$$

PROOF: This theorem of Rosenkrantz (195

Clearly $D(G)$ tion

$$(2.21) \quad T_n(t)f$$

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We pause to introduce $x \in [\delta, \infty)$ and 0 Thus $(G_n - G)T(s):$

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(2.16) we obtain

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LEMMA 2: Under the hypotheses of Theorem 3 there exists for any $\epsilon > 0$ a $\delta > 0$ such that

$$(2.17) \quad \limsup_{n \rightarrow \infty} E_x \left[\int_0^T I_{[0, \delta]}(Y_n(s)) ds \right] \leq \epsilon.$$

Setting aside the proof of (2.17) for the moment let us show that this implies strong convergence of the semi groups.

THEOREM 4: Under the hypotheses of Theorem 3

$$(2.18) \quad \lim_{n \rightarrow \infty} \|E_x(f(Y_n(t))) - E_x(f(Y(t)))\| = \lim_{n \rightarrow \infty} \|T_n(t)f(x) - T(t)f(x)\| = 0,$$

where the convergence is uniform for $t \in$ compact subsets of R^+ . Before proceeding to the proof of Theorem 4 we need a result due to the author, Rosenkrantz (1981), characterizing the domains $D(G_n)$ of the integro-differential operators G_n defined at (2.14).

LEMMA 3: Let G_n denote the strong infinitesimal generator of the normalized storage processes. Then

Case 1: If $r_n(x) = \bar{r}_n$, $x > 0$, $r_n(0) = 0$ we have

$$(2.19) \quad D(G_n) = \{f \in C_0^1(R^+) : f'(0) = 0\}$$

$$(2.20) \quad \text{Case 2: } D(G_n) = \{f \in C_0(R^+) : r_n(x)f'(x) \in C_0(R^+), \lim_{x \rightarrow 0} r_n(x)f'(x) = 0\}.$$

PROOF: This theorem is proved in exactly the same way as Theorem 4.6 on p. 219 of Rosenkrantz (1981). \square

Clearly $D(G) \subset D(G_n)$ and hence for every $f \in D(G)$ we have the representation

$$(2.21) \quad T_n(t)f(x) - T(t)f(x) = \int_0^t T_n(t-s)(G_n - G)T(s)f(x)ds,$$

cf. Burman (1979) p. 14, formula 2.2.

We pause to introduce some notation: If $g(x)$ is a function set $g_\delta(x) = g(x)$ if $x \in [\delta, \infty)$ and 0 otherwise and put $\bar{g}_\delta(x) = g(x) - g_\delta(x)$; so $g_\delta(x) + \bar{g}_\delta(x) = g(x)$. Thus $(G_n - G)T(s)f(x) = [(G_n - G)1(s)f]_\delta(x) + [(G_n - G)1(s)f]_{\bar{\delta}}(x)$ and therefore

$$\|T_n(t)f(x) - T(t)f(x)\| \leq \int_0^t \|[(G_n - G)1(s)f]_\delta(x)\| ds + \int_0^t \|T_n(t-s)[(G_n - G)T(s)f]_{\bar{\delta}}(x)\| ds$$

since $T_n(t)$ is a contraction semi group.

For $f \in D(G)$ the *a priori* estimate (A.8) and Lemma 1 together imply
 $\lim_{n \rightarrow \infty} (G_n - G)T(s)f(x) = 0$ uniformly on $[\delta, \infty)$ and uniformly in s , $0 \leq s \leq t$.
 Consequently $\lim_{n \rightarrow \infty} \int_0^t \|[(G_n - G)T(s)f]_\delta(x)\| ds = 0$. Similarly, $[(G_n - G)T(s)f]_\delta(x) \neq 0$
 only on the set $[0, \delta]$ and since by Lemma 1 and (A.8) $\|G_n T(s)f\|$ and $\|GT(s)f\|$
 are both uniformly bounded we conclude $\left| \int_0^t T_n(t-s)[(G_n - G)T(s)f]_\delta(x) ds \right| =$
 $\left| E_x \int_0^t [(G_n - G)T(s)f]_\delta(Y_n(t-s)) ds \right| \leq c' E_x \left(\int_0^t I_{[0, \delta]}(Y_n(s)) ds \right)$ where $c' =$
 $\sup_{n, 0 \leq s \leq t} \{ \|G_n T(s)f(x)\| + \|GT(s)f(x)\| \}$. We now apply Lemma 2 and choose δ so small
 that $\limsup_{n \rightarrow \infty} E_x \left(\int_0^t I_{[0, \delta]}(Y_n(s)) ds \right) \leq \varepsilon \cdot c^{-1}$ from which it follows at once that
 $\limsup_{n \rightarrow \infty} \|T_n(t)f(x) - T(t)f(x)\| \leq \varepsilon$ uniformly for t in compact subsets of R^+ . \square

We now turn to the proof of Lemma 3. Following Yamada let $\bar{Y}_n(t)$ denote the
 storage process with $\bar{r}_n(x) = \bar{r}_n$, $x > 0$ and $\bar{r}_n(0) = 0$. Since $\bar{r}_n(x) \geq r_n(x)$
 it is clear that $\bar{Y}_n(t) \geq Y(t)$ and in particular

$$E_x \left(\int_0^t I_{[0, \delta]}(Y_n(s)) ds \right) \leq E_x \left(\int_0^t I_{[0, \delta]}(\bar{Y}_n(s)) ds \right).$$

Thus to prove Lemma 3 it suffices to prove that

$$(2.22) \quad \limsup_{n \rightarrow \infty} E_x \left(\int_0^t I_{[0, \delta]}(\bar{Y}_n(s)) ds \right) \leq \varepsilon.$$

It is convenient to split the proof into two parts:

$$(2.23) \quad \lim_{n \rightarrow \infty} E_x \left(\int_0^t I_{[0]}(\bar{Y}_n(s)) ds \right) = 0$$

$$(2.24) \quad \limsup_{n \rightarrow \infty} E_x \left(\int_0^t I_{(0, \delta]}(\bar{Y}_n(s)) ds \right) \leq \varepsilon.$$

PROOF OF (2.23): The infinitesimal generator G'_n of $\bar{Y}_n(t)$ is

$$G'_n f(x) = -(\sqrt{n}/k_n) \bar{r}_n f'(x) + n\lambda \int_0^\infty [f(x+y) - f(x)] dH_n(y), \quad x > 0$$

$$G'_n f(0) = n\lambda \int_0^\infty [f(y) - f(0)] dH_n(y).$$

Applying Dynkin's formula as in Theorem 3.1 p. 216 of Rosenkrantz (1981),
 leads to the formula

$$(2.25) \quad E_x(\bar{Y}_n(t)) = x - (\sqrt{n}/k_n)(\bar{r}_n - \rho_n)t + (\sqrt{n}/k_n) \bar{r}_n E_x \left(\int_0^t I_{[0]}(Y_n(s)) ds \right).$$

In the appendix it will be shown that $\sup_{0 \leq s \leq t} E_x(\bar{Y}_n(s)) < \infty$ for every $t > 0$ and
 hence

$$(2.26) \quad (\sqrt{n}/k_n) \bar{r}_n$$

By (2.7) $\lim_{n \rightarrow \infty}$
 bounded whilst $\lim_{n \rightarrow \infty}$

$$(2.27) \quad E_x \left(\int_0^t I_{[0]}(Y_n(s)) ds \right)$$

Turning now to the
 and when $d = 0$.

Case 1: $d = \lim_{n \rightarrow \infty}$

LEMMA 4: For every
 the weak infinitesimal

PROOF: See Harrison
 $f_\alpha(x)$ is Lipschitz
 on $[0, \alpha]$. Thus,

$$(2.28) \quad \tilde{G}'_n f_\alpha(x) =$$

$$(2.29) \quad |\tilde{G}'_n f_\alpha(0)|$$

$$\tilde{G}'_n f_\alpha(x) \geq$$

Now for large
 on $(0, \alpha]$ provided

$$T_n(t)f_\alpha(x) - f_\alpha(x) =$$

other hand $E_x \left(\int_0^t \tilde{G}'_n f_\alpha(Y_n(s)) ds \right) =$

$$+ E_x \left(\int_0^t \tilde{G}'_n f_\alpha(\bar{Y}_n(s)) I_{[0]}(Y_n(s)) ds \right)$$

$$E_x \left(\int_0^t \tilde{G}'_n f_\alpha(\bar{Y}_n(s)) I_{[0]}(Y_n(s)) ds \right)$$

val $(0, \alpha]$ however,

$$\geq (d/2\alpha) E_x \left(\int_0^t I_{(0, \alpha]}(Y_n(s)) ds \right)$$

$$\limsup_{n \rightarrow \infty}$$

or imply

$$0 \leq s \leq t.$$

$$-G(T(s))\tilde{f}_\delta(x) \neq 0$$

$$\text{and } \|GT(s)f\|$$

$$|f]_\delta(x)ds| =$$

where $c' =$

pose δ so small

ows at once that

osets of \mathbb{R}^+ . \square

$\tilde{r}_n(t)$ denote the

$$\tilde{r}_n(x) \geq r_n(x)$$

$$(2.26) \quad (\sqrt{n}/k_n)\tilde{r}_n E_x \left[\int_0^t I_{[0]}(Y_n(s))ds \right] = E_x(\tilde{Y}_n(t)) - x + (\sqrt{n}/k_n)(\tilde{r}_n - r_n)t.$$

By (2.7) $\lim_{n \rightarrow \infty} (\sqrt{n}/k_n)(\tilde{r}_n - r_n)t = dt$ so the right hand side of (2.26) is bounded whilst $\lim_{n \rightarrow \infty} (\sqrt{n}/k_n)\tilde{r}_n = +\infty$, consequently

$$(2.27) \quad E_x \left[\int_0^t I_{[0]}(Y_n(s))ds \right] = O(n^{-1/2}).$$

Turning now to the proof of (2.24) we must consider separately the case when $d > 0$ and when $d = 0$.

Case 1: $d = \lim_{n \rightarrow \infty} (\sqrt{n}/k_n)(\tilde{r}_n - r_n) > 0$.

LEMMA 4: For every $\alpha > 0$ the function $f_\alpha(x) = [1 - (x/\alpha)]^+$ is in the domain of the weak infinitesimal generator \tilde{G}_n^* .

PROOF: See Harrison-Resnick (1976). Of course \tilde{G}_n^* is an extension of G_n^* and $f_\alpha(x)$ is Lipschitz continuous with $|f_\alpha(x+y) - f_\alpha(x)| \leq y \cdot \alpha^{-1}$, $f_\alpha'(x) = -\alpha^{-1}$ on $[0, \alpha]$. Thus,

$$(2.28) \quad \tilde{G}_n^* f_\alpha(x) = (\sqrt{n}/k_n)\tilde{r}_n \alpha^{-1} + n\lambda \int_0^\alpha [f_\alpha(x+y) - f_\alpha(x)]dH_n(y), \quad 0 < x \leq \alpha,$$

$$(2.29) \quad |\tilde{G}_n^* f_\alpha(0)| \leq (\sqrt{n}/k_n)\tilde{r}_n \cdot \alpha^{-1}, \quad \tilde{G}_n^* f_\alpha(x) = 0, \quad x > \alpha. \quad \text{In particular}$$

$$\begin{aligned} \tilde{G}_n^* f_\alpha(x) &\geq (\sqrt{n}/k_n)\tilde{r}_n \cdot \alpha^{-1} - n\lambda \int_0^\alpha \alpha^{-1} \cdot y dH_n(y) \\ &= \alpha^{-1}(\sqrt{n}/k_n)(\tilde{r}_n - r_n) \quad \text{on } (0, \alpha]. \end{aligned}$$

Now for large n , $(\sqrt{n}/k_n)(\tilde{r}_n - r_n) \geq d/2 > 0$ and this implies $\tilde{G}_n^* f_\alpha(x) \geq d/2\alpha$ on $(0, \alpha]$ provided n is large enough. Notice that $|f_\alpha(x)| \leq 1$ and hence

$$T_n(t)I_\alpha(x) - f_\alpha(x) = \int_0^t T_n(s)\tilde{G}_n^* f_\alpha(x)ds \quad \text{implies} \quad |E_x \left[\int_0^t f_\alpha(\tilde{Y}_n(s))ds \right]| \leq 2. \quad \text{On the}$$

$$\text{other hand } E_x \left[\int_0^t \tilde{G}_n^* f_\alpha(\tilde{Y}_n(s))ds \right] = E_x \left[\int_0^t \tilde{G}_n^* f_\alpha(\tilde{Y}_n(s))I_{[0]}(\tilde{Y}_n(s))ds \right]$$

$$+ E_x \left[\int_0^t \tilde{G}_n^* f_\alpha(\tilde{Y}_n(s))I_{(0, \alpha]}(\tilde{Y}_n(s))ds \right]. \quad \text{From (2.27) and (2.29) we see at once that}$$

$$E_x \left[\left| \int_0^t \tilde{G}_n^* f_\alpha(\tilde{Y}_n(s))I_{[0]}(\tilde{Y}_n(s))ds \right| \right] \text{ is bounded, by } M \text{ say, as } n \rightarrow \infty. \quad \text{On the inter-}$$

$$\text{val } (0, \alpha] \text{ however, } \tilde{G}_n^* f_\alpha(x) \geq (d/2\alpha) \text{ and therefore } \left| \int_0^t \tilde{G}_n^* f_\alpha(\tilde{Y}_n(s))I_{(0, \alpha]}(\tilde{Y}_n(s))ds \right|$$

$$\geq (d/2\alpha)E_x \left[\int_0^t I_{(0, \alpha]}(\tilde{Y}_n(s))ds \right]. \quad \text{Therefore as } n \rightarrow \infty \text{ we get}$$

$$\limsup_{n \rightarrow \infty} E_x \left[\int_0^t I_{(0, \alpha]}(\tilde{Y}_n(s))ds \right] \leq (2 + M)2\alpha/\alpha.$$

$$_n(y), \quad x > 0$$

rantz (1981),

$$(Y_n(s))ds \Big\}.$$

every $t \geq 0$ and

The proof is now completed by choosing $\epsilon \leq d/(4 + 2M)$.

Case 2: $d = 0$. In this case $\lim_{n \rightarrow \infty} G_n^t f(x) = (1/2)f''(x)$ for every $f \in D(G) =$

$C_0^1(\mathbb{R}^+; f(0) = 0)$ i.e., the limit process in this case is reflecting Brownian motion $|w(t)|$. Thus the original Trotter-Kato theorem itself implies that

$\lim_{n \rightarrow \infty} \|E_x(f(\tilde{Y}_n(t)) - f(|w(t)|))\| = 0$. It is a consequence of a theorem of Aldous

(1978) that $\tilde{Y}_n(t)$ converges weakly to $|w(t)|$ or if one prefers, the weak convergence may be deduced from a more general result due to Kurtz (1981), Theorem 4.4.

It is well known that reflecting Brownian motion has a local time $\alpha(t, y, \omega)$ and therefore $\int_0^t 1_{[0, \cdot]}(|w(s)|) ds = \int_0^t \alpha(t, y, \cdot) dy \leq t$, where $\alpha(t, y, \cdot)$ is jointly continuous in (t, y) for each ω . By Lebesgue's dominated convergence theorem then

we have $\lim_{\delta \rightarrow 0} E \left(\int_0^\delta \alpha(t, y, \cdot) dy \right) = 0$ and so given any $\epsilon > 0$ $\exists \delta > 0$ such that

$$E \left(\int_0^\delta 1_{[0, \cdot]}(|w(s)|) ds \right) < \epsilon.$$

Let us denote by P_n and P the measures induced on $D[0, T]$ by the $\tilde{Y}_n(t)$ and $|w(t)|$ processes respectively. It is well known that the functional

$\omega \mapsto \int_0^t 1_{[0, \cdot]}(\omega(s)) ds$, here ω is a path in $D[0, T]$, is continuous almost every-

where with respect to the measure P , cf. Billingsley (1968), pp. 230-231. This fact together with the weak convergence of P_n to P and Theorem (5.2iii) p. 31 of Billingsley, op. cit., imply

$$(2.30) \quad \lim_{n \rightarrow \infty} E_x \left(\int_0^\delta 1_{[0, \cdot]}(\tilde{Y}_n(s)) ds \right) = E_x \left(\int_0^\delta 1_{[0, \cdot]}(|w(s)|) ds \right) < \epsilon. \quad \square$$

The proof of Theorem 4 is now complete.

APPENDIX

Let $Bf(x) = \int_0^x f(y) dy$ acting on the domain $D(B)$ (1971) that B acts continuous, contraction. Estimate was also obtained a more general result:

LEMMA: For every $f \in D(B)$ such that

$$(A.1) \quad \|f'\| \leq M$$

We next observe that $f \in D(B)$ defined by (2.15) and clearly $D(C) \supset D(B)$.

THEOREM: There exists the inequality

$$(A.2) \quad \|Cf\| \leq M$$

REMARK: When (A.2) respect to B - see

PROOF: Let $\|f\|_{[a, b]}$

$\sup_k \|f\|_{[k, k+1]}$ where k The proof of inequality (1.13).

$$(A.3) \quad \|f'\|_{[a, b]}$$

Specializing (A.3) to

$$(A.4) \quad \|f'\|_{[k, k+1]}$$

If now $f \in C_0^2(\mathbb{R})$ hence

$$(A.5) \quad \|f'\|_{[k, k+1]}$$

APPENDIX

Let $Bf(x) = (1/2)f''(x) + (\gamma/x)f'(x)$, $\gamma > -(1/2)$ denote the Bessel operator acting on the domain $D(B) = \{f \in C_0^2(\mathbb{R}^+); f'(0) = 0\}$. It was shown in Brezis, et al. (1971) that B acting on $D(B)$ generates a positivity preserving, strongly continuous, contraction semi group $T_1(t): C_0(\mathbb{R}^+) \rightarrow C_0(\mathbb{R}^+)$. The following *A priori* estimate was also obtained (see Theorem (A.1) p. 411 of Brezis et al. (1971), where a more general result is given):

LEMMA: For every $f \in D(B)$ there exists a constant $\epsilon > 0$, depending only on γ , such that

$$(A.1) \quad \|f''\| \leq \epsilon \|Bf\|.$$

We next observe that the operator $Gf = Bf + Cf$ where B is Bessel operator defined by (2.15) and $Cf = -df'$, i.e., G is a perturbation of the operator B ; clearly $D(C) \supset D(B)$.

THEOREM: There exist constants $a > 0$, $0 < b < 1/2$ such that for every $f \in D(B)$ the inequality

$$(A.2) \quad \|Cf\| \leq a \|f''\| + b \|Bf\|, \text{ holds.}$$

REMARK: When (A.2) holds the operator C is said to be *relatively bounded with respect to* B - see Kato (1976), p. 190.

PROOF: Let $\|g\|_{[a,b]} = \sup_{a \leq x \leq b} |g(x)|$ and observe that, for $g \in C_0^1(\mathbb{R}^+)$, $\|g\| = \sup_k \|g\|_{[k,k+1]}$ where the sup is taken over all non-negative integers $k = 0, 1, 2, \dots$. The proof of inequality (A.3) below is to be found in Kato, op. cit. p. 192, formula (1.13).

$$(A.3) \quad \|f''\|_{[a,b]} \leq [(b-a)/(n+2)] \cdot \|f'''\|_{[a,b]} + [2(n+1)/(b-a)] \|f'\|_{[a,b]}$$

for every $f \in C^2[a,b]$ and every $n \geq 1$.

Specializing (A.3) to the special case $[a,b] = [k,k+1]$ yields

$$(A.4) \quad \|f''\|_{[k,k+1]} \leq (n+2)^{-1} \|f'''\|_{[k,k+1]} + 2(n+1) \|f'\|_{[k,k+1]}.$$

If now $f \in C_0^2(\mathbb{R}^+)$ we have $\|f'''\|_{[k,k+1]} \leq \|f'''\|$ and $\|f'\|_{[k,k+1]} \leq \|f'\|$ and hence

$$(A.5) \quad \|f''\|_{[k,k+1]} \leq (n+2)^{-1} \|f'''\| + 2(n+1) \|f'\|.$$

Consequently for every $f \in D(B)$ we have

$$\|f'\| = \sup_k \|f'\|_{[k, k+1]} \leq (n+2)^{-1} \|f'\| + 2(n+1) \|f\| \quad \text{and in particular} \\ (A.6) \quad \|Gf\| \leq d(n+2)^{-1} \|f'\| + 2d(n+1) \|f\| \leq d(n+2)^{-1} \|Bf\| + 2d(n+1) \|f\|$$

where we used (A.1) in the last step.

Thus by choosing $n > 2\beta d - 2$ we have $\beta d(n+2)^{-1} < \frac{1}{2}$ and this completes the proof (A.2) with $a = 2d(n+1)$. \square

The following a priori estimate is also an easy consequence of the above calculation:

$$(A.7) \quad \|f''\| \leq 2\|Gf\| + 4\beta d(n+1) \|f\|.$$

PROOF: Since $Bf = Gf - Cf$ we have from (A.1) and (A.6) that $\|f''\| \leq \beta\|Gf\| + \beta\|Cf\| \leq \beta\|Gf\| + \beta d(n+2)^{-1} \|f'\| + 2\beta d(n+1) \|f\|$. Since $\beta d(n+2)^{-1} < \frac{1}{2}$ we have $(1 - \beta d(n+2)^{-1}) \|f''\| \leq \beta\|Gf\| + 2\beta d(n+1) \|f\|$ and hence $\|f''\| \leq 2\beta\|Gf\| + 4\beta d(n+1) \|f\|$. \square

Combining all these estimates together with Theorem 2.7 of Kato p. 501 we arrive at the

THEOREM: The operator $G = B + C$ generates a positivity preserving, strongly continuous contraction semi group $T(t): C_0(R^+) \rightarrow C_0(R^+)$ with domain $D(G) = D(B) = \{f \in C_0^2(R^+): f'(0) = 0\}$. Moreover for every $f \in D(G)$ we have the following a priori estimate: $\|f''\| \leq 2\beta\|Gf\| + 4\beta d(n+1) \|f\|$. In particular if $f \in D(G)$ then $T(s)f \in D(G)$ and therefore

$$(A.8) \quad \|(\partial/\partial x^2)T(s)f(x)\| \leq 2\beta\|GT(s)f\| + 4\beta d(n+1) \|f\| \\ \leq 2\beta\|T(s)Gf\| + 4\beta d(n+1) \|f\| \\ \leq 2\beta\|Gf\| + 4\beta d(n+1) \|f\|.$$

We have used the facts that $T(s)$ commutes with its infinitesimal generator G and that $T(s)$ is a contraction. Notice that the right hand is independent of s .

We next turn our attention to deriving the estimate:

$$(A.9) \quad \sup_{0 \leq s \leq t} E_x(\bar{Y}_n(s)^2) \leq x^2 + t.$$

This clearly implies $\sup_{0 \leq s \leq t} E_x(\bar{Y}_n(s)) < \infty$ which is all we needed to derive (2.27).

PROOF OF (A.9): Let

$$G_n^1 U(t, x)$$

thus $[(GU)(t) + G_n^1 U(t, x)]$ is a supermartingale. \square

PROOF OF (A.9): Let $U(t, x) = x^2 - t$ and observe that

$$\begin{aligned} G'_n U(t, x) &= -2(\sqrt{n}/k_n) \bar{r}_n x + n \lambda_n \int_0^\infty (2xy + y^2) dH_n(y) \\ &= 1 - (2\sqrt{n}/k_n) x (\bar{r}_n - \rho_n) \leq 1 \quad \text{on } \mathbb{R}^+. \end{aligned}$$

Thus $[(\partial U / \partial t) + G'_n] U(t, x) = -1 + G'_n U(t, x) \leq 0$; consequently $\bar{Y}_n(t) - t^2$ is a supermartingale. Thus $E_x(\bar{Y}_n(t)^2 - t) \leq x^2$ or $E_x(\bar{Y}_n(t)^2) \leq x^2 + t$. \square

since $\beta d(n+2)^{-1} < \frac{1}{2}$

ratio p. 501 we

ving, strongly con-

$D(G) = D(B) =$

the following

If $f \in D(G)$ then

al generator G

independent of s .

to derive (2.27).

ACKNOWLEDGEMENT

Research supported by the U.S. Air Force of Scientific Research under Grant 82-0167.

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4. PERFORMING ORGANIZATION REPORT NUMBER(S)		5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR. 85-0056	
6a. NAME OF PERFORMING ORGANIZATION University of Massachusetts	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research	
6c. ADDRESS (City, State and ZIP Code) Department of Mathematics & Statistics GRC Tower, Amherst MA 01003		7b. ADDRESS (City, State and ZIP Code) Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332-6448	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR	8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-82-0167	
8c. ADDRESS (City, State and ZIP Code) Bolling AFB DC 20332-6448		10. SOURCE OF FUNDING NOS.	
		PROGRAM ELEMENT NO. 61102F	TASK NO. A5
		PROJECT NO. 2304	WORK UNIT NO.
11. TITLE (Include Security Classification) WEAK CONVERGENCE OF A SEQUENCE OF QUEUEING AND STORAGE PROCESSES TO A SINGULAR DIFFUSION			
12. PERSONAL AUTHOR(S) Walter A. Rosenkrantz			
13a. TYPE OF REPORT Technical	13b. TIME COVERED FROM _____ TO _____	14. DATE OF REPORT (Yr., Mo., Day) NOV 84	15. PAGE COUNT 15
16. SUPPLEMENTARY NOTATION			
17. COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
FIELD	GROUP	SUB. GR.	
19. ABSTRACT (Continue on reverse if necessary and identify by block number) It has been known for a long time that heavy traffic limit theorems in queueing theory are but a special case of the so-called diffusion approximation in Physics and Genetics. Take for example Kingman's (1962) heavy traffic approximation for the stationary waiting time distribution for a sequence of GI/GI/1 queues $Q(\alpha)$ depending on a parameter α . Denote the waiting time, excluding service, of the n^{th} customer by $W(n, \alpha)$ and let $U(n, \alpha) = S(n, \alpha) - T(n, \alpha)$ where $S(n, \alpha)$ = service time of the n^{th} customer and $T(n, \alpha)$ = inter arrival time between the n^{th} and $(n+1)^{\text{st}}$ customer and assume $E(U(n, \alpha)) = -\alpha\sigma$, variance of $U(n, \alpha) = \sigma^2$, $\alpha > 0$. Then we have the following Theorem 1 (Kingman (1962)): $\lim_{n \rightarrow \infty} P((\alpha/\sigma)W(n, \alpha) \leq x) = 1 - \exp(-2x), \quad 0 \leq x < \infty, \quad \text{provided} \quad \lim_{n \rightarrow \infty} \alpha^2 n = \infty.$ Somewhat later Kingman (1965) presented a more elegant but heuristic proof of this result which justifies referring to such a theorem as a diffusion approximation. It is worthwhile sketching the heuristic proof of Theorem 1 here, referring			
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS <input type="checkbox"/>		21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
22a. NAME OF RESPONSIBLE INDIVIDUAL MAJ Brian W. Woodruff		22b. TELEPHONE NUMBER (Include Area Code) (202) 767- 5027	22c. OFFICE SYMBOL NM

(CONTINUED)

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ITEM #19, ABSTRACT, CONTINUED:

the reader to Rosenkrantz (1980) for a rigorous proof as well as an estimate of the rate of convergence. To begin with, one notes that (1.1) $F_{n,\alpha}(x) = P((\alpha/\sigma)W(n,\alpha) \leq x) = P(\sup_{0 \leq t \leq \alpha n} y_{n,\alpha}(t) \leq x)$ where $y_{n,\alpha}(t)$ is a certain stochastic process with continuous paths. One can then show, formally at least, that (1.2) $\lim_{n \rightarrow \infty, \alpha \rightarrow 0} y_{n,\alpha}(t) = y(t)$ where $y(t) = w(t) - t$. Here $w(t)$ is the standard 1-dimensional Wiener process and so $y(t)$ is the Wiener process with negative drift. It follows at once from (1.2) that (1.3) $\lim_{n \rightarrow \infty, \alpha \rightarrow 0} P(\sup_{0 \leq t \leq \alpha n} y_{n,\alpha}(t) \leq x) = P(\sup_{0 \leq t < \infty} y(t) \leq x)$ and an easy calculation, see e.g. Karlin-Taylor (1975), p.361, yields the result that $P(\sup_{0 \leq t < \infty} y(t) \leq x) = 1 - \exp(-2x)$, $0 \leq x < \infty$. Another and simpler example of a heavy traffic limit theorem is the following: let $N_n(t)$ denote the queue size of an M/M/1 queue with arrival rate λ_n , mean service time distribution μ_n^{-1} and traffic intensity $\rho_n = \lambda_n/\mu_n$. Assume $\lambda_n = \mu_n - \delta n^{-1/2}$ for some $\delta > 0$, so $0 < \rho_n < 1$ and denote by σ_n^2 the variance of the service time distribution which in this case equals μ_n^{-2} .

THEOREM 2: Assume $\lambda = \lim_{n \rightarrow \infty} \lambda_n = \lim_{n \rightarrow \infty} \mu_n = \mu$ so $\lim_{n \rightarrow \infty} \rho_n \uparrow 1$, and $\lim_{n \rightarrow \infty} \sigma_n^2 = \sigma^2$; then $\lim_{n \rightarrow \infty} N_n(nt)/\sqrt{n} = y(t)$ where $y(t)$ is the Wiener process on $R^+ = [0, \infty)$ with variance $\lambda + \sigma^2 \mu^3$, negative drift δ and reflected at the origin. Theorem 2 has been extended in many ways and by many authors including Iglehart and Whitt. The survey article by Whitt (1974) is a useful reference for the reader interested in these developments.

In each of the heavy traffic limit theorems cited above the limit process has turned out to be the Wiener process with a negative drift satisfying, where appropriate, a reflecting boundary condition. Recently Yamada (1982) has given a diffusion approximation for a sequence of storage processes $X_n(t)$ where the limit process $Y(t)$ is no longer a Wiener process with a negative drift but is instead a Bessel process with negative drift. This result is of more than routine interest. It shows for example that the set of possible limit processes that can occur in queueing and storage theory is a much larger class than Theorems 1, 2 and the survey article by Whitt (1974) would lead us to believe existed. In addition Yamada's theorem (a precise version of which will be stated below as Theorem 3) offers a challenge to the traditional methods by which such limit theorems are usually proved. In particular, neither the Trotter-Kato-Kurtz method of Kurtz (1969) nor the martingale method of Papnicolaou, Stroock and Varadhan (1977) are directly applicable to this limit theorem because of some nontrivial technical problems of independent interest and the solutions of which are also of independent interest. It is the purpose of this paper to give a new and simpler proof of Yamada's theorem using some results due to Brezis, Rosenkrantz and Singer, with an appendix by P. D. Lax, (1971) which, restated in the more modern terminology of today, implies that the martingale problem for the operator corresponding to the Bessel process with drift has a unique solution - see Stroock-Varadhan (1979) and Ikeda-Watanabe (1981) for a general discussion of these ideas. It turns out however that the estimates we needed to make the martingale methods work already imply the strong convergence of the semigroups in the sense of Trotter-Kato - see Theorem 4 below. These as well as other results from Functional Analysis are collected in an appendix. We shall also use the standard notations: $C_0(R^+) = \{f: f \text{ bounded and continuous on } R^+ = [0, \infty) \text{ and } \lim_{x \rightarrow \infty} f(x) = 0\}$, $f^{(l)}(x) = l^{\text{th}} \text{ derivative of } f$, $C_0^k(R^+) = \{f \in C_0(R^+): f^{(l)} \in C_0(R^+), 1 \leq l \leq k\}$. We make $C_0(R^+)$ into a Banach space in the usual way by giving it the norm $\|f\| = \sup_{0 \leq x < \infty} |f(x)|$. The symbol \blacksquare denotes the end of a proof.

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